ON THE UNIQUENESS OF $\mathbb{C}^*$-ACTIONS ON AFFINE SURFACES

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Abstract. It is an open question whether every normal affine surface V over $\mathbb{C}$ admits an effective action of a maximal torus $\mathbb{T} = \mathbb{C}^{*n}$ ($n \le 2$) such that any other effective $\mathbb{C}^*$-action is conjugate to a subtorus of $\mathbb{T}$ in $\mathrm{Aut}(V)$. We prove that this holds indeed in the following cases: (a) the Makar-Limanov invariant $\mathrm{ML}(V) \ne \mathbb{C}$ is nontrivial, (b) V is a toric surface, (c) $V = \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$, where Δ is the diagonal, and (d) $V = \mathbb{P}^2 \backslash Q$, where Q is a nonsingular quadric. In case (a) this generalizes a result of Bertin for smooth surfaces, whereas (b) was previously known for the case of the affine plane (Gutwirth [Gut]) and (d) is a result of Danilov-Gizatullin [DG] and Doebeli [Do].

1. Introduction

The classification problem for reductive group actions on affine spaces or, more generally, on affine varieties, has a long history. By [Kam, KP, KKMR, Po] any reductive group action on $\mathbb{A}^2_{\mathbb{C}}$ and $\mathbb{A}^3_{\mathbb{C}}$ is conjugate to a linear one. The same holds for connected reductive groups acting on $\mathbb{A}^4_{\mathbb{C}}$ except possibly for $\mathbb{C}^*$ and $\mathbb{C}^{*2}$ [Pa, Po] (cf. also [BaHab]), and for tori $\mathbb{T}^n$ acting effectively on $\mathbb{A}^n_{\mathbb{C}}$, $\mathbb{A}^{n+1}_{\mathbb{C}}$ [BB] and on the affine toric n-folds [De, Gub]. According to [Sc, MMP, Kn] many finite nonabelian groups and any connected reductive nonabelian group admit a non-linearizable action on some affine space $\mathbb{A}^n_{\mathbb{C}}$. In the local case the existence of a maximal reductive subgroup of $\mathrm{Aut}(V, 0)$, which contains a conjugate of any other connected reductive subgroup, was established in [HM]. In [DG, Do] the same was shown to be true for the smooth affine quadric surface in $\mathbb{A}^3_{\mathbb{C}}$. It is an open question whether this holds as well for every normal affine surface V. In this paper we give some partial positive results, see Corollary 5.5 below.

Bertin's Theorem [Be, Corollary 2.3] asserts that, for a smooth affine surface V non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, which admits a minimal compactification V by a simple normal crossing divisor D with a non-linear dual graph Γ_D, any two effective $\mathbb{C}^*$-actions on V are conjugated in the automorphism group $\mathrm{Aut}(V)$. On the other hand, by Gizatullin's Theorem [Gi, Theorems 2 and 3] (see also [BML, Be] or [Du] for the more general case of normal surfaces), if $V \not\cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$ then Γ_D is linear if and only if the Makar-Limanov invariant

$$\mathrm{ML}(V) := \bigcap_{\partial \in \mathrm{LND}(A)} \ker \partial$$

of V is trivial that is, $\mathrm{ML}(V) = \mathbb{C}$, where $\mathrm{LND}(A)$ stands for the set of all locally nilpotent derivations of the coordinate ring $A = H^0(V, \mathcal{O}_V)$ of V. The latter holds if

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and only if V admits two non-equivalent effective $\mathbb{C}_+$-actions i.e., two $\mathbb{C}_+$-actions with different general orbits (see e.g., [FZ_2]).

We present here an alternative proof of Bertin's Theorem, valid more generally for normal affine surfaces. Our proof is not based on the properties of completions and so is independent of Gizatullin's Theorem. In Theorem 3.3 below we show that, as soon as $\mathrm{ML}(V) \neq \mathbb{C}$ and $V \not\cong \mathbb{C}^* \times \mathbb{C}^*$ or $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$, any two effective $\mathbb{C}^*$-actions on V are conjugated via an element of a $\mathbb{C}_+$-subgroup of the automorphism group $\mathrm{Aut}(V)$.

For a surface V with an effective action of the 2-torus $\mathbb{T}$ we prove in Theorem 4.5 that any effective $\mathbb{C}^*$-action on V is conjugated in $\mathrm{Aut}(V)$ to the action of a subtorus of $\mathbb{T}$. In the case of the affine plane $V = \mathbb{A}^2_{\mathbb{C}}$ this gives another proof of the classical Gutwirth Theorem [Gut] saying that the linearization conjecture for $\mathbb{C}^*$-actions on $\mathbb{A}^n_{\mathbb{C}}$ holds in dimension 2.

In Section 5 we deduce similar results for the surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$ and $\mathbb{P}^2 \backslash Q$, where Δ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ and Q is a nonsingular quadric in $\mathbb{P}^2$. We show that any two effective $\mathbb{C}^*$-actions on one of these surfaces are conjugate in the automorphism group. According to a result of Gizatullin and Popov (see [FZ_2, Theorem 4.12]) these are the only normal affine surfaces that admit a nontrivial $\mathbf{SL}_2$-action except for the affine plane and the affine Veronese cones over the rational normal curves. We note that by [FZ_2, Proposition 4.14] any two $\mathbf{SL}_2$-actions on a normal affine surface are conjugate in the automorphism group.

Our interest in such kind of results is related with our studies [FZ_1, FZ_2] on the Dolgachev-Pinkham-Demazure (or DPD, for short) presentation of a normal affine surface V endowed with a $\mathbb{C}^*$-action. We show in Corollary 4.3 that for surfaces with a non-trivial Makar-Limanov invariant, except in the case of the surfaces $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathbb{A}^1_C \times \mathbb{C}^*$, this DPD-presentation is uniquely determined up to a natural equivalence. For affine toric surfaces we describe in Section 4 the possible ambiguities in the choice of a DPD-presentation. We also deduce the uniqueness of the DPD-presentation for the surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$ and $\mathbb{P}^2 \backslash Q$ as above, which are non-toric and have a trivial Makar-Limanov invariant.

In Corollary 5.5 we deduce that all maximal connected reductive subgroups of the automorphism group $\mathrm{Aut}(V)$ are conjugate, and any connected reductive subgroup is contained in a maximal one, besides the remaining open case when the surface is non-toric and has a trivial Makar-Limanov invariant. In the forthcoming paper [FKZ] we will solve this remaining case, up to one exception, by showing that the automorphism group $\mathrm{Aut}(V)$ contains a unique class of conjugated $\mathbb{C}^*$-subgroups, which also implies the uniqueness of a DPD-presentation up to a natural equivalence.

2. Preliminaries

Let k be an algebraically closed field of characteristic 0. We recall the following definition.

Definition 2.1. Let A be a k-algebra, not necessarily associative or commutative. A derivation $\delta : A \to A$ is called *locally bounded* if every finite subset of A is contained in some δ-invariant linear subspace $V \subseteq A$ of finite dimension over k.

For instance, if $\delta : A \to A$ is a semisimple derivation on A i.e., A has a k-basis consisting of eigenvectors of δ then δ is locally bounded.

Clearly if δ is locally bounded then A is the union of its finite dimensional δ-invariant subspaces V. In particular, the restriction of δ to each such subspace V admits a unique Jordan-Chevalley decomposition

$$\delta = \delta_s + \delta_n,$$

where δ_s and δ_n are the semisimple and the nilpotent parts of δ, respectively. The restriction to a smaller δ-invariant subspace respects this decomposition, hence δ_s and δ_n are well-defined k-linear maps on A that are again locally bounded. We will see in Lemma 2.2 below that these maps are as well derivations on A.

For $\alpha \in k$ we consider the linear subspace

$$A_\alpha := \bigcup_{i\ge 0} \ker(\delta - \alpha\,\mathrm{id})^i\,.$$

From standard linear algebra we know that

$$V = \bigoplus_\alpha (V \cap A_\alpha)\,,$$

whenever V is a finite dimensional δ-invariant subspace of A. Thus A is a graded vector space

$$A = \bigoplus_{\alpha\in k} A_\alpha\,.$$

Clearly, δ_s and δ_n leave invariant every subspace A_α. Moreover δ_s acts on A_α via multiplication by α, whereas $\delta_n|A_\alpha$ is locally nilpotent. More precisely, we have the following lemma (see e.g., [Ch, Thm. 16] or [Ja, Ch. II, Ex. 8] for the case of algebras of finite dimension, [SW] for complete algebras, and also [CD, 2.1]).

Lemma 2.2. (a) *A is a graded algebra, i.e. $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for all $\alpha,\ \beta \in k$;*
(b) *δ_s and δ_n are again derivations on A;*
(c) *δ_s is homogeneous and acts on A_α via multiplication by α, so is automatically locally bounded, whereas δ_n is homogeneous of degree* 0 *and locally nilpotent.*

Proof. (a) For homogeneous elements $x \in A_\alpha$ and $y \in A_\beta$ we have

$$(\delta - (\alpha+\beta)\,\mathrm{id})(xy) = x(\delta - \beta\,\mathrm{id})(y) + (\delta - \alpha\,\mathrm{id})(x)y\,,$$

and hence by induction

$$(\delta - (\alpha+\beta)\,\mathrm{id})^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (\delta - \alpha\,\mathrm{id})^i(x)\cdot(\delta - \beta\,\mathrm{id})^{n-i}(y)\,.$$

It follows that $(\delta-(\alpha+\beta)\,\mathrm{id})^n(xy)$ vanishes for $n \gg 0$, and so $xy \in A_{\alpha+\beta}$, proving (a).

By definition δ_s acts via multiplication with α on A_α. Since A is a graded algebra this shows that δ_s is a degree 0 homogeneous derivation on A and so is $\delta_n = \delta - \delta_s$. Now (b) and (c) follow. □

In the next lemma we consider the set

$$M := \{\alpha \in k | A_\alpha \neq 0\}\,,$$

and we let $\mathbb{N}M$ and $\mathbb{Z}M$ be the additive subsemigroup and the subgroup of k, respectively, generated by M.

Lemma 2.3. *If A is a finitely generated k-algebra then*

(a) *also* $\mathbb{N}M$ *and* $\mathbb{Z}M$ *are finitely generated, and*
(b) A *admits an effective action of a torus* $(k^*)^r$, *where* $r := \mathrm{rk}_{\mathbb{Z}}\mathbb{Z}M$.

Proof. The proof of (a) is elementary and we omit it. To show (b) we note that $A = \bigoplus_{\alpha\in M} A_\alpha$ is graded by the semigroup $\mathbb{N}M$, which is a subsemigroup of $\mathbb{Z}M \cong \mathbb{Z}^r$. An effective action of $(k^*)^r$ on A is then given by

$$(\lambda_1, \dots, \lambda_r).x_\alpha := \lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r} x_\alpha, \quad x_\alpha \in A_\alpha,$$

where $(\lambda_1, \dots, \lambda_r) \in (k^*)^r$ and $\alpha \in M \subseteq \mathbb{Z}^r$ has components $\alpha_1, \dots, \alpha_r \in \mathbb{Z}$. □

Lemma 2.4. *If* $\partial : A \to A$ *is a locally nilpotent derivation then*

$$\exp(-\partial)\delta\exp(\partial) = \delta + [\delta, \partial] \tag{1}$$

for any derivation $\delta : A \to A$ *satisfying*

$$[[\delta, \partial], \partial] = 0\,. \tag{2}$$

Proof. We denote $\Delta_i := [\delta, \partial^i]$. Then (2) says that $[\Delta_1, \partial] = 0$. By the Jacobi identity we get

$$[\Delta_i, \partial] = [\Delta_1, \partial^i]\,, \tag{3}$$

and so by (2) $[\Delta_i, \partial] = 0 \quad \forall i \geq 1$.

Assuming by induction that, for a given i,

$$\Delta_i = i\partial^{i-1}\Delta_1\,, \tag{4}$$

we obtain from the above equalities:

$$\Delta_{i+1} = [\delta, \partial^{i+1}] = \Delta_i\partial + \partial^i\Delta_1 = i\partial^{i-1}\Delta_1\partial + \partial^i\Delta_1 = (i+1)\partial^i\Delta_1\,,$$

which proves (4) for all $i \geq 1$. Thus we get

$$\delta\partial^i = \Delta_i + \partial^i\delta = \partial^i\delta + i\partial^{i-1}\Delta_1\,,$$

and consequently

$$\begin{aligned} \delta\exp(\partial) &= \textstyle\sum_{i=0}^\infty \delta\partial^i/i! \\ &= \left(\textstyle\sum_{i=0}^\infty \partial^i/i!\right)\delta + \left(\textstyle\sum_{i=1}^\infty (\partial^{i-1}/(i-1)!)\right)\Delta_1 \\ &= \exp(\partial)(\delta + [\delta, \partial]). \end{aligned}$$

Multiplying with $\exp(-\partial)$ from the left gives (1). □

3. Main theorem

In this section we formulate and prove our main results. For any $\mathbb{Z}$-graded finitely generated $\mathbb{C}$-algebra

$$A = \bigoplus_{i\in\mathbb{Z}} A_i\,,$$

every derivation ∂ of A has a unique decomposition $\partial = \sum_{i=k}^{l} \partial_i$, where $\partial_i : A \to A$ is a homogeneous derivation of degree i. The proof of the following simple lemma is left to the reader.

Lemma 3.1. *If* $\partial = \sum_{i=k}^l \partial_i$ *is locally bounded then* ∂_l *and* ∂_k *are also locally bounded. In particular, if* $l > 0$ $(k < 0)$ *then* ∂_l *(respectively,* ∂_k*) is locally nilpotent.*

3.2. In the sequel we let A be the coordinate ring of a normal affine surface with a $\mathbb{C}^*$-action so that $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is graded. The infinitesimal generator of this $\mathbb{C}^*$-action is a semisimple derivation δ on A that acts via $\delta(a) = \deg(a)\cdot a$ for a homogeneous element $a \in A$. As was shown in [FZ$_2$, Proposition 2.4], for a homogeneous locally nilpotent derivation $\partial \neq 0$ the derivation

$$\exp(-\partial)\delta\exp(\partial)$$

is again semisimple and defines a $\mathbb{C}^*$-action which is, in general, different from the given one. Conversely, we have the following result.

Theorem 3.3. *If* $\mathrm{ML}(A) \neq \mathbb{C}$, $\mathrm{Spec}(A) \not\cong \mathbb{C}^*\times\mathbb{C}^*$ *and* $\not\cong \mathbb{A}^1_{\mathbb{C}}\times\mathbb{C}^*$ *then every semisimple derivation* $\tilde\delta$ *on* A *is of the form*

$$\tilde\delta = c\cdot\exp(-\partial)\delta\exp(\partial), \qquad c\in\mathbb{C},$$

for some locally nilpotent derivation ∂ *on* A. *Consequently, any two effective* $\mathbb{C}^*$*-actions of* A, *after possibly switching one of them by the automorphism* $\lambda \longmapsto \lambda^{-1}$ *of* $\mathbb{C}^*$, *are conjugate via an automorphism of* A *provided by a* $\mathbb{C}_+$*-action on* A *and, moreover, coincide whenever* $ML(A) = A$.

The latter assertion leads to the following corollary.

Corollary 3.4. *If a normal affine surface* $V = \mathrm{Spec}(A)$, *non-isomorphic to* $\mathbb{C}^*\times\mathbb{C}^*$, *admits two effective* $\mathbb{C}^*$*-actions with infinitesimal generators* $\delta, \tilde\delta$, *where* $\delta \neq \pm\tilde\delta$, *then it also admits a non-trivial* $\mathbb{C}_+$*-action.*

To prove Theorem 3.3 we need a few preparations.

3.5. We suppose below that $\mathrm{ML}(A) \neq \mathbb{C}$, $\mathrm{Spec}(A) \not\cong \mathbb{C}^*\times\mathbb{C}^*$ and $\not\cong \mathbb{A}^1_{\mathbb{C}}\times\mathbb{C}^*$. If A admits a homogeneous locally nilpotent derivation ∂ of degree zero then by Lemma 3.8 and Corollary 3.28 in [FZ$_2$] either $A \cong \mathbb{C}[t,u]$ or $A \cong \mathbb{C}[t,u,u^{-1}]$ with $t\in A_0, u\in A_d$ homogeneous and $\partial = \partial/\partial t$, which is excluded by our assumptions. According to Corollary 3.27(i) and Theorem 4.5 from [FZ$_2$], either all nontrivial homogeneous locally nilpotent derivations on A are of positive degree, or all of them are of negative degree. By switching the grading to the opposite one, if necessary, we may suppose in the sequel that A does not admit a homogeneous locally nilpotent derivation of degree ≤ 0.

Lemma 3.6. *With the assumptions of 3.2 and 3.5, for every nonzero locally nilpotent derivation* ∂ *of* A *the following hold.*

(a) ∂ *is a linear combination of commuting homogeneous locally nilpotent derivations of strictly positive degrees.*
(b) *The derivations* $[\delta,\partial]$ *and* ∂ *commute.*
(c) $\exp(-\partial)\delta\exp(\partial) = \delta + [\delta,\partial]$.
(d) *There exists a locally nilpotent derivation* ∂' *on* A *such that* $\partial = [\delta,\partial']$.

Proof. (a) Let us write ∂ as a sum of homogeneous derivations

$$\partial = \sum_{i=k}^{l}\partial_i \quad\text{with}\quad \partial_k,\partial_l\neq 0.$$

Since clearly ∂_k and ∂_l are again locally nilpotent (see e.g., [Re]), by our convention in 3.5 above we have $k \geq l > 0$. Moreover, since $\mathrm{ML}(A)\neq\mathbb{C}$, ∂ and ∂_k are equivalent,

so define equivalent $\mathbb{A}^1_{\mathbb{C}}$-fibrations, and $\partial = a\partial_l$ for some $a \in \mathrm{Frac}(\ker \partial_l)$ (see e.g., [FZ$_2$, Lemma 4.5]). It follows that the ∂_i are commuting locally nilpotent derivations, proving (a).

Now (b) follows from (a) since $[\delta, \partial] = \sum_{i=k}^{l} i\partial_i$, and (c) follows from Lemma 2.4 by virtue of (b). Finally, (d) can be deduced by taking $\partial' := \sum_{i=k}^{l} \partial_i/i$. □

Proof of Theorem 3.3. For a semisimple derivation $\tilde{\delta}$ of A, we consider its decomposition $\tilde{\delta} = \sum_{i=k}^{l} \partial_i$ into homogeneous components with ∂_k, $\partial_l \neq 0$. By Lemma 3.1, if $k < 0$ then ∂_k is locally nilpotent, which is excluded by our convention in 3.5. Thus $l \geq k \geq 0$.

Now the proof proceeds by induction on l. If $l = 0$ then $\tilde{\delta} = \partial_0$ is semisimple, homogeneous of degree 0 and commutes with δ. Thus δ and ∂_0 are equal up to a constant factor. Indeed, otherwise $V := \mathrm{Spec}(A)$ would be a toric surface non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$, hence $\mathrm{ML}(A) = \mathbb{C}$ (see Example 2.8 in [FZ$_2$]), which contradicts our assumption. Clearly, the constant factor above is equal to ± 1 as soon as both δ and ∂_0 generate effective $\mathbb{C}^*$-actions.

Assume now that $l > 0$. By Lemma 3.6(d) we find a locally nilpotent derivation ∂' with $[\tilde{\delta}, \partial'] = \partial_l$, and so by Lemma 3.6(c)

$$\exp(\partial')\tilde{\delta}\exp(-\partial') = \tilde{\delta} - [\tilde{\delta}, \partial'] = \tilde{\delta} - \partial_l.$$

Thus $\tilde{\delta} - \partial_l$ is again semisimple and its homogeneous components have degrees $\leq l - 1$. Applying the induction hypothesis, the result follows. □

4. Toric surfaces and uniqueness of a DPD-presentation

4.1. Let A be a normal 2-dimensional $\mathbb{C}$-algebra with a grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$ associated to an effective $\mathbb{C}^*$-action. We recall that such a grading admits a DPD-presentation as follows (see [FZ$_1$]).

Elliptic case: Here $A_0 = \mathbb{C}$, and up to switching the grading we have $A_- := \bigoplus_{i<0} A_i = 0$. The curve $C := \mathrm{Proj}\, A$ is normal and carries a $\mathbb{Q}$-divisor D of positive degree unique up to linear equivalence such that

$$A = A_0[D] := \bigoplus_{i\geq 0} H^0(C, \mathcal{O}_C(\lfloor iD \rfloor))u^i \subseteq \mathrm{Frac}(C)[u]\,. \tag{5}$$

Parabolic case: Here $A_- = 0$, but A_0 is 1-dimensional and so defines a smooth curve $\mathrm{Spec}\, A_0 = \mathrm{Proj}\, A$. As before C carries a $\mathbb{Q}$-divisor D, now of arbitrary degree and again unique up to linear equivalence such that (5) holds.

Hyperbolic case: This case is characterized by $A_+, A_- \neq 0$. The subrings $A_{\geq 0} := \bigoplus_{i\geq 0} A_i$ and $A_{\leq 0}$ are parabolic and as before admit presentations $A_{\geq 0} = A_0[D_+] \subseteq \mathrm{Frac}(C)[u]$ and $A_{\leq 0} = A_0[D_-] \subseteq \mathrm{Frac}(C)[u^{-1}]$, respectively. Thus A is the subring

$$A = A_0[D_+, D_-] := A_0[D_+] + A_0[D_-] \subseteq \mathrm{Frac}(C)[u, u^{-1}].$$

Moreover by Theorem 4.3 in [FZ$_1$] $D_+ + D_- \leq 0$, and the pair (D_+, D_-) is determined uniquely by the graded algebra A up to a linear equivalence

$$(D_+, D_-) \sim (D'_+, D'_-) :\Leftrightarrow (D_+, D_-) = (D'_+ + \mathrm{div} f, D'_- - \mathrm{div} f) \quad \text{with} \quad f \in \mathrm{Frac}(C)^\times\,.$$

The question arises whether a DPD-presentation is determined uniquely, up to the linear equivalence as above, an automorphism of C and in the hyperbolic case by an interchange of D_+ and D_-, by the geometry of the surface V alone, disregarding the choice of a $\mathbb{C}^*$-action. In Corollary 4.3 below we show that, indeed, this is the case at least for surfaces with a non-trivial Makar-Limanov invariant.

4.2. Let us recall [FZ$_2$, Theorem 4.5] that the Makar-Limanov invariant of a surface $V = \operatorname{Spec}(A)$, where $A = A_0[D_+, D_-]$ with $D_+ + D_- \le 0$ and $D_+ + D_- \ne 0$, is trivial if and only if $A_0 \cong \mathbb{C}[t]$ and the fractional parts $\{D_\pm\}$ of $D_\pm$ are zero or are concentrated at one point, say, $p_\pm \in \mathbb{A}^1_{\mathbb{C}}$. If still $A_0 \cong \mathbb{C}[t]$, but the second condition holds for precisely one of the divisors $D_\pm$ then $ML(A) = \mathbb{C}[x]$ for a nonzero homogeneous element $x \in A$. Otherwise $ML(A) = A$ that is, A does not admit non-zero locally nilpotent derivations.

From Theorem 3.3 above and Theorem 4.3 in [FZ$_1$] we derive the following corollary.

Corollary 4.3. *For a non-toric normal affine surface $V = \operatorname{Spec}(A)$ the following hold.*

(a) If V admits an effective elliptic $\mathbb{C}^*$-action then this $\mathbb{C}^*$-action is unique up to the automorphism $\lambda \longmapsto \lambda^{-1}$ of $\mathbb{C}^*$. In particular a DPD-presentation $A = A_0[D]$ is unique up to linear equivalence of D.

(b) If V admits a parabolic $\mathbb{C}^*$-action then every $\mathbb{C}^*$-action on V is parabolic. Moreover, if $A = A_0[D] = A_0'[D']$ are DPD-presentations corresponding to two $\mathbb{C}^*$-actions then there is an isomorphism $\varphi : C \to C'$ of the curves defined by A_0 and A_0', respectively, such that D and $\varphi^*(D')$ are linearly equivalent.

(c) If $\mathrm{ML}(A) \ne \mathbb{C}$ and

$$A = A_0[D_+, D_-] \quad \text{and} \quad A = A_0'[D_+', D_-']$$

are DPD-presentations of A corresponding to two hyperbolic $\mathbb{C}^*$-actions on A then there is an isomorphism $\varphi : A_0 \to A_0'$ such that the pair of $\mathbb{Q}$-divisors (D_+, D_-) on the curve $C = \operatorname{Spec}(A_0)$ is linearly equivalent to one of the pairs $\varphi^*(D_+', D_-')$ or $\varphi^*(D_-', D_+')$.

Proof. If in case (a) there is a second $\mathbb{C}^*$-action on A not related to the first one by an automorphism of $\mathbb{C}^*$ then by Lemma 3.1 there is a locally nilpotent derivation on A and so by Theorem 3.3 in [FZ$_2$] V is toric contrary to our assumption.

(b) and (c) follow immediately from the fact that any two $\mathbb{C}^*$-actions on V are conjugate in $\operatorname{Aut}(V)$ by Theorem 3.3. □

4.4. We note that toric surfaces $V = \operatorname{Spec}(A)$ admit many non-conjugated $\mathbb{C}^*$-actions given by non-conjugated one-parameter subgroups of the 2-torus $\mathbb{T}^2 = \mathbb{C}^* \times \mathbb{C}^*$. It also has many distinct DPD-presentations, up to permuting D_+ and D_- and to linear equivalence. Any pair of divisors $D_\pm = \mp\frac{e_\pm}{d_\pm}[0]$ with $e_\pm, d_\pm \in \mathbb{Z}$, $d_+ > 0$, $d_- < 0$, $\gcd(e_\pm, d_\pm) = 1$ and $(D_+ + D_-)(0) < 0$ defines a toric surface $V = \operatorname{Spec}(A_0[D_+, D_-])$, where $A_0 \cong \mathbb{C}[t]$. Two such toric surfaces $V = \operatorname{Spec}(A_0[D_+, D_-])$ and $V' = \operatorname{Spec}(A_0[D_+', D_-'])$ are isomorphic if and only if the sublattices $\mathbb{Z}(e_+, d_+) + \mathbb{Z}(e_-, d_-)$ and $\mathbb{Z}(e_+', d_+') + \mathbb{Z}(e_-', d_-')$ of $\mathbb{Z}^2$ are equivalent up to the action of the group of integer matrices with determinant ± 1 (see [De, Gub] for a more general result, or also the proof of Theorem 4.15 in [FZ$_1$]). According to Theorem 4.5 below this is the only ambiguity in the choice of a DPD-presentation for V.

In the rest of this section we concentrate on affine toric surfaces. We remind the reader that any such surface $V = \mathrm{Spec}(A)$ is isomorphic to $\mathbb{C}^{*2}$, $\mathbb{A}^1_C \times \mathbb{C}^*$ or to

$$V_{d,e} = \mathrm{Spec}\, A_{d,e} := \mathrm{Spec}(\mathbb{C}[X,Y]^{\mathbb{Z}_d}) = \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_d,$$

where the cyclic group $\mathbb{Z}_d = \langle \zeta \rangle$ of dth roots of unity acts on the polynomial ring $\mathbb{C}[X,Y]$ via

$$\zeta.X = \zeta X \quad \text{and} \quad \zeta.Y = \zeta^e Y$$

for some e satisfying $0 \le e < d$ and $\gcd(e,d) = 1$. The standard action of the 2-torus $\mathbb{T}' := \mathbb{C}^{*2}$ on $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[X,Y])$ commutes with the action of $\mathbb{Z}_d$ and so descends to $V_{d,e}$. Dividing out the kernel $K \cong \mathbb{Z}_d$ gives an effective action of the 2-torus $\mathbb{T} = \mathbb{T}'/K$ on V.

We note that for $ee' \equiv 1 \mod d$ the surfaces $V_{d,e}$ and $V_{d,e'}$ are isomorphic via the map induced by the morphism

$$\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}} \quad \text{with} \quad (x,y) \mapsto (y,x).$$

This morphism is φ-equivariant, where φ is the map of $\mathbb{T}' = \mathbb{C}^* \times \mathbb{C}^*$ interchanging the factors. Moreover, $V_{d,e}$ and $V_{d',e'}$ are isomorphic as normal surfaces if and only if

$$d = d' \quad \text{and either} \quad e = e' \quad \text{or} \quad ee' \equiv 1 \mod d, \tag{6}$$

see e.g. [FZ$_1$, Example 2.3].

Every 1-dimensional subgroup of $\mathbb{T}$ isomorphic to $\mathbb{C}^*$ provides a $\mathbb{C}^*$-action on V. Conversely we have the following result.

Theorem 4.5. *If $V = \mathrm{Spec}(A)$ is an affine toric surface and if $\mathbb{C}^*$ acts effectively on V then this action is conjugate to the action of a subtorus of $\mathbb{T}$.*

Proof. In the case $V = \mathbb{T}$ this is evident. So we can assume in the sequel that $V \neq \mathbb{T}$. This $\mathbb{C}^*$-action then defines a grading $A = \bigoplus_{i \in Z} A_i$ of A. As V is toric and not a torus it admits a $\mathbb{C}_+$-action. Our assertion is an immediate consequence of the following claim.

Claim. Either $A = \mathbb{C}[z, v, v^{-1}]$ for homogeneous elements z, v of A, or there is a graded isomorphism $A \cong A_{d,e}$, where the grading on $A_{d,e}$ is provided by a subtorus of $\mathbb{T}$ as above.

In the case of an elliptic grading this is just Theorem 3.3 in [FZ$_2$]. If the grading is parabolic and if there is a horizontal $\mathbb{C}_+$-action, i.e. the general orbits of the $\mathbb{C}_+$-action map dominantly to $V/\mathbb{C}^*$, then this is a consequence of Theorem 3.16 in [FZ$_2$].

In the case that the grading is parabolic and there is only a vertical but no horizontal $\mathbb{C}_+$-action, we have $A \cong A_0[u]$ with $\deg u = 1$ and $A_0 = \mathbb{C}[t, t^{-1}]$. In fact, the curve $C = \mathrm{Spec}(A_0)$ is either $\mathbb{A}^1_C$ or $\mathbb{C}^*$, since the open orbit of the torus action on V maps dominantly onto C. If $C \cong \mathbb{A}^1_{\mathbb{C}}$ then $A = A_0[D]$ with $A_0 \cong \mathbb{C}[t]$ and the fractional part of D is supported on at least two points, since by our assumption there is no horizontal $\mathbb{C}_+$-action on V (cf. [FZ$_2$, Theorem 3.16]). By Proposition 3.8 in [FZ$_1$] this would imply that V has at least two singular points, which is impossible.

Therefore $C \cong \mathbb{C}^*$ and so $A_0 \cong \mathbb{C}[t, t^{-1}]$. As there is no dominant morphism $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{C}^*$ and thus also no dominant morphism $V_{d,e} \to \mathbb{C}^*$, we have $V \cong \mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$. In particular V is smooth, and $A = A_0[D]$ with D being an integral (so, principal) divisor. Thus $D \sim 0$ proving the claim in this case.

In the remaining case the grading on A is hyperbolic so that $A = A_0[D_+, D_-]$. Since V admits a $\mathbb{C}_+$-action, we have necessarily $A_0 \cong \mathbb{C}[t]$ and $C = \mathbb{A}^1_{\mathbb{C}}$, see [FZ$_2$, Corollary 3.23]. The Picard group of a toric surface is trivial and so by Corollary 4.24 in [FZ$_1$] we have $l \le 1$, where

$$l := \operatorname{card}\,\{p \in C : D_+(p) + D_-(p) < 0\}.$$

If $l = 0$ then $D_+ = -D_-$ and so by Remark 4.5 in [FZ$_1$], A contains a unit of nonzero degree. As V admits a $\mathbb{C}_+$-action then by Corollary 3.27 in [FZ$_2$] we have $A = \mathbb{C}[z, v, v^{-1}]$ with homogeneous elements $z, v \in A$, which proves our claim in this case.

If $l = 1$ then by Corollary 4.24 in [FZ$_1$] the orbit map $V \to C = V//\mathbb{C}^*$ has no irreducible multiple fiber and the divisor $D_+ + D_-$ is concentrated in one point, say, $p \in \mathbb{A}^1_{\mathbb{C}}$. Since $D_\pm(a) \in \mathbb{Z}$ and $D_+(a) + D_-(a) = 0$ for every point $a \ne p$, the pair (D_+, D_-) is equivalent to a pair

$$\left(-\frac{e_+}{d_+}[p], \frac{e_-}{d_-}[p]\right).$$

As in the proof of Theorem 4.15 in [FZ$_1$] there exist integers d, e with $0 \le e < d$ and $\gcd(d, e) = 1$ so that $A \cong A_{d,e}$ as graded rings, where the grading on $A_{d,e}$ is defined by a subgroup of $\mathbb{T}$ isomorphic to $\mathbb{C}^*$. This finishes the proof. $\square$

As a particular case we obtain the following classical result.

Corollary 4.6. (Gutwirth [Gut]) *Every $\mathbb{C}^*$-action on $\mathbb{A}^2_{\mathbb{C}}$ is linearizable.*

Remarks 4.7. 1. Any two effective actions of a 2-torus on a normal affine surface V are conjugate. This follows from the fact that any toric surface is equivariantly isomorphic to one of the surfaces $\mathbb{C}^{*2}$, $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{C}^*$ or $V_{d,e}$, and two such surfaces are isomorphic as abstract surfaces if and only if (6) holds.

2. Every normal affine surface $V = \operatorname{Spec}(A)$ with an elliptic or parabolic $\mathbb{C}^*$-action and with a trivial Makar-Limanov invariant is toric (see Theorems 3.3 and 3.14 in [FZ$_2$]). This yields that there can be at most one parabolic DPD-presentation $A = A_0[D]$ on an affine normal surface, up to the equivalence described in Corollary 4.3(b).

5. Homogeneous affine surfaces

In this section we show the following result.

Theorem 5.1. *Let V be one of the surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$ or $\mathbb{P}^2 \backslash Q$, where Δ is the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ and Q is a nonsingular quadric in $\mathbb{P}^2$. Then any two $\mathbb{C}^*$-actions on V are conjugate in* $\operatorname{Aut}(V)$.

In case $V = \mathbb{P}^2 \backslash Q$ this is a result of [DG], see also [Do]. This theorem follows immediately from the following two more general statements.

Proposition 5.2. *Every smooth Gorenstein $\mathbb{C}^*$-surface $V = \operatorname{Spec}(A)$ with $\operatorname{Cl}(V) \cong \mathbb{Z}$ and $\operatorname{ML}(V) = \mathbb{C}$ is isomorphic to $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$, where Δ is the diagonal. Moreover, there is a graded isomorphism $A \cong A_0[D_+, D_-]$, where $A_0 = \mathbb{C}[t]$, $D_+ = 0$ and $D_- = -[1] - [-1]$.*

Proof. An elliptic or parabolic $\mathbb{C}^*$-surface with trivial Makar-Limanov invariant is toric by Theorems 3.3 and 3.16 from [FZ$_2$], and so its Picard group vanishes. Thus under

our assumptions the $\mathbb{C}^*$-action on V is necessarily hyperbolic and provides a DPD-presentation $A = A_0[D_+, D_-]$ with $A_0 = \mathbb{C}[t]$ and a pair of $\mathbb{Q}$-divisors $D_\pm$ on the affine line $\mathbb{A}^1_\mathbb{C} = \operatorname{Spec}\mathbb{C}[t]$ such that $D_+ + D_- \le 0$, see Corollary 3.23 in [FZ$_2$]. As the divisor class group of A is isomorphic to $\mathbb{Z}$ by Theorem 4.22 in [FZ$_1$] we have

$$l = \operatorname{card}\{p \in \mathbb{A}^1_\mathbb{C} : D_+(p) + D_-(p) < 0\} = 2.$$

Thus there are unique points $p_1, p_2 \in \mathbb{A}^1_\mathbb{C}$ with $D_+(p_i) + D_-(p_i) < 0$ for $i = 1, 2$. Moreover there can be no multiple fiber of the projection $V \to \mathbb{A}^1_\mathbb{C}$. Indeed, by *loc. cit.* every such fiber contributes to the torsion of $\operatorname{Cl}(A)$. Therefore we may assume that

$$D_+(p_i) = -\frac{e_i^+}{m_i^+} \quad \text{and} \quad D_-(p_i) = \frac{e_i^-}{m_i^-}$$

with $m_i^+ > 0$ and $m_i^- < 0$ and $\gcd(e_i^\pm, m_i^\pm) = 1$ for $i = 1, 2$. Since V is smooth, we have

$$\begin{vmatrix} e_i^+ & m_i^+ \\ e_i^- & m_i^- \end{vmatrix} = -1, \tag{7}$$

see [FZ$_1$, Theorem 4.15]. In this case using again Theorem 4.22 from [FZ$_1$] the divisor class group is generated by the orbit closures $O_i^\pm$ over the points p_i modulo the relations

$$M_1 = M_2 = 0 \quad \text{and} \quad E_1 + E_2 = 0,$$

where

$$M_i := m_i^+[O_i^+] - m_i^-[O_i^-] \quad \text{and} \quad E_i := e_i^+[O_i^+] - e_i^-[O_i^-].$$

By assumption the canonical divisor of V is trivial, and by Corollary 4.25 in [FZ$_1$] it is given by

$$K_V = \sum_{i=1}^{2}((m_i^+ - 1)[O_i^+] + (-m_i^- - 1)[O_i^-]) \sim -\sum_{i=1}^{2}([O_i^+] + [O_i^-]).$$

Thus the divisor $K = -\sum([O_i^+] + [O_i^-])$ is contained in the subgroup generated by M_1, M_2 and $E_1 + E_2$, and so the determinant $\det(K, M_1, M_2, E_1 + E_2)$ vanishes. Using (7) this leads to the relation

$$m_1^+ + m_1^- = m_2^+ + m_2^-. \tag{8}$$

Since the Makar-Limanov invariant of V is trivial, the fractional part of each of the divisors $D_\pm$ is concentrated in one (possibly the same) point, see 4.2. Thus we may assume that $m_1^+ = 1$, and $m_i^- = -1$ for at least one value $i = 1, 2$. In the case $m_1^- = -1$ equation (8) shows that $m_2^+ + m_2^- = 0$. By (7) this yields $m_2^+ = -m_2^- = 1$. Similarly, if $m_2^- = -1$ then the right hand side in (8) is ≥ 0 whereas the term on the left is ≤ 0. This forces $m_2^+ = 1$ and $m_1^- = -1$.

Hence the divisors $D_\pm$ are necessarily integral. Replacing them by an equivalent pair of divisors we can achieve that $D_+ = 0$ and $D_- = -[p_1] - [p_2]$. After performing an automorphism of $\mathbb{A}^1_\mathbb{C}$ we can also assume that $p_1 = 1$ and $p_2 = -1$ and so the DPD-presentation has the required form. Now the isomorphism of V and $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$ follows from Example 5.1 in [FZ$_1$]. □

Proposition 5.3. *A smooth* $\mathbb{C}^*$*-surface* $V = \operatorname{Spec} A$ *with* $\operatorname{Pic}(V) \cong \mathbb{Z}_2$ *and* $\operatorname{ML}(V) = \mathbb{C}$ *is isomorphic to* $\mathbb{P}^2 \backslash Q$*, where* Q *is a smooth quadric in* $\mathbb{P}^2$*. Moreover there is a graded isomorphism* $A \cong A_0[D_+, D_-]$*, where* $A_0 = \mathbb{C}[t]$*,* $D_+ = \frac{1}{2}[0]$ *and* $D_- = -\frac{1}{2}[0] - [1]$*.*

Proof. With the same arguments as in the proof of Proposition 5.2 the given $\mathbb{C}^*$-action is necessarily hyperbolic and V admits a DPD-presentation $V = \mathrm{Spec}(A_0[D_+, D_-])$ with $A_0 = \mathbb{C}[t]$. As V is smooth, by Theorem 4.22 in [FZ$_1$] the condition $\mathrm{Pic}(V) = \mathbb{Z}_2$ gives that $l = 1$ and there is only one irreducible multiple fiber of multiplicity 2. Thus we may suppose that this multiple fiber lies over 0 and then the fractional parts are

$$\{D_\pm(0)\} = 1/2.$$

The fractional parts of $D_\pm$ at all other points must be zero since $\mathrm{ML}(A) = \mathbb{C}$ (see Theorem 4.5 in [FZ$_2$]). Hence after passing to an equivalent pair of divisors, we may suppose that $D_+ = \frac{1}{2}[0]$ and then $D_- = -\frac{1}{2}[0] - a[p]$ for some $a \in \mathbb{N}$ and $p \in \mathbb{A}^1_{\mathbb{C}}$. After performing an automorphism of $\mathbb{A}^1_{\mathbb{C}}$, if nedeed, we may also suppose that $p = 1$. As V is smooth we obtain from Theorem 4.15 in [FZ$_1$] that necessarily $a = 1$. Thus

$$A = A_0[D_+, D_-] \quad \text{with} \quad D_+ = \frac{1}{2}[0] \quad \text{and} \quad D_- = -\frac{1}{2}[0] - [1].$$

This proves the second assertion. The first one follows by comparing with Example 5.1 from [FZ$_2$]. ☐

Remark 5.4. If a torus of dimension k acts effectively on a variety V of dimension n, then $k \le n$. Thus the rank of a reductive group G acting effectively on V is bounded by n. As every increasing chain of connected reductive groups with bounded rank becomes stationary, this shows the following fact: *every connected reductive subgroup of* $\mathrm{Aut}(V)$ *is contained in a maximal one.*

In the Introduction we posed the question whether any two maximal connected reductive subgroups of the automorphism group of a normal affine surface are conjugate. We note that this would follow if one could prove that

($*$) any two effective $\mathbb{C}^*$-actions on a non-toric surface V with $\mathrm{ML}(V) = \mathbb{C}$ are conjugate.

In fact, if V admits no $\mathbf{SL}_2$-action then every maximal connected reductive subgroup of $\mathrm{Aut}(V)$ is a torus. In this case the result follows from ($*$), Theorems 3.3, 4.5 and Remark 4.7(1).

If V admits an action of $\mathbf{SL}_2$ then by the Theorem of Gizatullin and Popov mentioned in the Introduction V is one of the surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$, $\mathbb{P}^2 \backslash Q$, or $V_{d,1}$ ($d \ge 1$).

If V is one of the surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$ or $\mathbb{P}^2 \backslash Q$ then the standard actions of $\mathbf{PGL}_2$ on V cannot be extended to a larger connected reductive group, since otherwise there would be an action of a 2-torus on V. Thus in these cases the maximal connected reductive subgroups of $\mathrm{Aut}(V)$ are isomorphic to $\mathbf{PGL}_2$ and, moreover, any two such subgroups are conjugate in $\mathrm{Aut}(V)$ by Proposition 4.14 in [FZ$_2$].

Similarly, if $V = V_{d,1}$ then $\mathbf{GL}_2/\mathbb{Z}_d$ is a maximal connected reductive subgroup of $\mathrm{Aut}(V)$. Given another maximal connected reductive subgroup G of $\mathrm{Aut}(V)$ we may suppose by Remark 4.7(1) that its maximal torus is equal to the standard one in $\mathbf{GL}_2/\mathbb{Z}_d$. Now it is easy to see that G and $\mathbf{GL}_2/\mathbb{Z}_d$ are equal (alternatively one can apply Lemma 4.17 in [FZ$_2$]).

From this remark and Theorems 3.3, 4.5 and 5.1 we deduce the following corollary.

Corollary 5.5. *For a normal affine surface V, any two maximal connected reductive subgroups of the automorphism group* $\mathrm{Aut}(V)$ *are conjugate in* $\mathrm{Aut}(V)$ *except possibly*

in the case where V is non-toric, has a trivial Makar-Limanov invariant and is not isomorphic to one of the surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$ or $\mathbb{P}^2 \backslash Q$.

The forthcoming paper [FKZ] will be devoted to this remaining case.

Added in proofs. We are grateful to Peter Russell who showed us an example, given any $k \geq 2$, of a non-toric smooth affine surface with a trivial Makar-Limanov invariant that admits k mutually non-conjugated $\mathbb{C}^*$-subgroups in the automorphism group. This gives, in general, a negative answer to the question in the Introduction. Such a surface appears as the complement in a Hirzebruch surface of a section with selfintersection number $k + 1$. These surfaces were studied in [DG, II].

We are also grateful to Jürgen Hausen who informed us, after appearing of the first e-print version of this paper, that in [BeHau] an n-dimensional generalization of our Theorem 4.5 had been proven. Namely, it was shown that an effective action of the torus $\mathbb{T}^{n-1}$ on an n-dimensional affine toric variety V is conjugate in the automorphism group $\mathrm{Aut}(V)$ to a subgroup of the standard action of $\mathbb{T}^n$ on V.

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